SPREADING OF A FILM OF NONLINEARLY VISCOUS LIQUID OVER A HORIZONTAL SMOOTH SOLID SURFACE

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The equations and self-similar solutions of the problem of the spreading of a thin film of non-Newtonian liquid over a horizontal plane with nonlinear variation in liquid flow over time are analyzed. The case of an infinite plane and a narrow gap between vertical walls are considered.

1. Spreading of a Thin Film over an Infinite Plane

In describing the flow in films of many materials – such as dye lasers, petroleum films, thin layers of polymer solutions and melts, extruding magma, etc. – the non-Newtonian properties of these media must be taken into account. For slow flows, the dependence of the viscosity on the rate of shear is of most importance. Below, analysis is confined to the role of this non-Newtonian feature of real liquids (neglecting the compressibility).

The problem of spreading of a thin film may be simplified in the presence of a small parameter $\varepsilon \equiv H/L$, the ratio of the characteristic film thickness H to the characteristic horizontal scale L. Reducing the spatial coordinates, the horizontal and vertical velocity components, the time, the amount by which the pressure exceeds the atmospheric value, and the viscosity to dimensionless form by means of the scale factors H, L, U, εU, L/U, gH, and μ, the incompressibility condition (unit density is assumed) and dynamic equations may be written in the following dimensionless form

$$\nabla \mathbf{u} + \frac{\partial w}{\partial z} = 0,$$

$$\varepsilon \operatorname{Re} \frac{d\mathbf{u}}{dt} = -\omega \nabla p + \frac{\partial}{\partial z}\left(\eta \frac{\partial \mathbf{u}}{\partial z}\right) + O(\varepsilon^2),$$

$$\varepsilon^2 \operatorname{Re} \frac{dw}{dt} = -\omega\left(\frac{\partial p}{\partial z} + 1\right) + O(\varepsilon^2),$$

$$\eta = \eta\left(\left|\frac{\partial \mathbf{u}}{\partial z}\right|\right) + O(\varepsilon^2),\ \operatorname{Re} \equiv \frac{UH}{\nu},\ \omega \equiv \frac{gH^3}{\nu LU}.$$

The determining rheological equation adopted here is the simplest common generalization of the nonlinear one-dimensional relation between the shear stress and the shear rate $\tau = \eta(\gamma)\gamma$, $\gamma = f(\tau)$, a generalization in which the stress and deformation-rate tensors are coaxial, and the proportionality factor – the viscosity – depends only on the quadratic invariant of the deformation-rate tensor [1].

Suppose that at a smooth solid surface z = 0 adhesion conditions apply: $\mathbf{u} = 0$, $w = 0$ (i.e., slip is not taken into account). At the free surface $z = h(x, t)$, continuity of the normal (taking account of surface tension) and tangential stress, together with the kinematic condition, leads to the following dimensionless relations

$$p_a - p = \operatorname{We} \nabla^2 h + O(\varepsilon^2),\ \operatorname{We} \equiv \frac{\sigma}{gL^2},$$

$$\tau + O(\varepsilon^2) = 0,\ \tau = \eta\left(\left|\frac{\partial \mathbf{u}}{\partial z}\right|\right)\frac{\partial \mathbf{u}}{\partial z},\ \frac{\partial h}{\partial t} + \mathbf{u}\nabla h = w,\ (z = h).$$

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Here τ denotes the horizontal shear-stress vector.

Consideration is confined to the first approximation, omitting all terms of the equations with the small parameter ε (including those with the product $\varepsilon \mathrm{Re}$). In this noninertial approximation ($\varepsilon \mathrm{Re} \ll 1$), the motive force is the horizontal pressure gradient due to the horizontal difference in film thickness, and is balanced (when $\omega \sim 1$) by viscous friction. The flow rate and the shear-stress vector are expressed here in terms of the height of the surface h(x, t) (below, only the dimensional variables denoted by the same symbols are used)

$$\mathbf{u} = [hF_0(\tau_w) - (h - z) F_0(\tau)] \boldsymbol{\tau}/\tau,$$

$$w = \nabla \{(h - z)\mathbf{u} - [h^2 F_1(\tau_w) - (h - z)^2 F_1(\tau)] \boldsymbol{\tau}/\tau\},$$

$$\boldsymbol{\tau} = -(h - z) \nabla (gh - \sigma \nabla^2 h), \quad \boldsymbol{\tau}_w \equiv \boldsymbol{\tau}|_{z=0},$$

$$F_l(\tau) \equiv \int_0^1 d\zeta\, \zeta^l f(\tau\zeta), \quad \tau = |\boldsymbol{\tau}|,$$

whereas the function h(x, t) describing the position of the unknown free boundary satisfies the equation

$$\frac{\partial h}{\partial t} = -\nabla \left(h^2 F_1(\tau_w) \frac{\boldsymbol{\tau}_w}{\tau_w} \right), \quad \boldsymbol{\tau}_w = -h\nabla (gh - \sigma\nabla^2 h). \tag{1}$$

Here and below, $F_\ell(\tau)$ denotes the integral moments of the rheological function $f(\tau)$, and τ_w denotes the tangential stress at the horizontal surface.

Taking into account that the contributions of the gravitational force and the surface tension on the right-hand side of Eq. (1) are not additive in the case of a nonlinearly viscous liquid, the surface tension will be neglected below, assuming large films (We $\ll$ 1). Then the simplified equation for h differs only in notation from that derived in [2].

A remarkable property of this nonlinear equation is the existence of solutions that remain finite for any finite time. Analysis of such solutions in the vicinity of the edge of the spreading film is possible in sufficiently general form. In the course of a short time interval, a small section of the boundary may be regarded as plane and moving at constant velocity v_0. Then the derivative $\partial/\partial t$ is replaced by $-v_0\partial/\partial x$ and the equation for h may readily be integrated once: $v_0 \approx hF_1(\tau_w)$, $\tau_w = -gh\partial h/\partial x$.

In the model approximation of a nonlinear non-Newtonian liquid ($0 < n < \infty$) and a Newtonian viscous liquid, in particular ($n = 1$, $k = \eta_0$)

$$f(\tau) = \left(\frac{\tau}{k}\right)^{1/n}, \quad F_l(\tau) = \frac{n}{1 + n(1 + l)} f(\tau),$$

the equation for h may readily be integrated a second time in explicit form. As a result, the following expressions are obtained close to the edge of the film (when $x_0 - x \ll x_0$; $h = 0$ when $x \geq 0$)

$$h^{n+2} \approx (n + 2) \frac{\varkappa}{g} (x_0 - x), \quad \tau_w \approx \frac{\varkappa}{h^n}, \quad \varkappa \equiv k \left(\frac{2n + 1}{n} v_0\right)^n, \tag{2}$$

i.e., the free surface at the edge of the film forms a 90° angle with the horizontal substrate (neglecting the surface tension, Van der Waals forces, etc., but then the given asymptotes are only meaningful at some distance from the edge; immediately at the edge, special consideration is required because ε is large), and the shear stress increases on approaching the edge. The film thickness at the edge changes more sharply for dilatational liquids ($n > 1$) and more smoothly for pseudoplastic materials ($n < 1$) than for a Newtonian viscous liquid ($n = 1$). The smoothest form $(x_0 - x)^{1/2}$ corresponds to the limit as $n \to 0$. and the limit as $n \to \infty$ corresponds to a film of uniform thickness with a discontinuity at $x = x_0$.

In describing the flow of real liquids, a power law may only be satisfactory in a limited range of stress (and velocity). However, since large stress is attained close to the edge of the film, it may be applicable, for example, for such liquids as highly viscous melts and solutions with large polymer concentrations. On the other hand, at large shear stress, in many non-Newtonian liquids (dilute polymer solutions, suspensions of solid particles, etc.), conditions of "second Newtonian" viscosity η_∞ may be attained (in contrast to highly viscous liquids), so that the form of the film at the edge is also "Newtonian" – Eq. (2) with n = 1, k = η_∞ = const. In some cases, the form of the film may be expected to reflect the appearance of both asymptotes simultaneously, with the formation of a "Newtonian precursor" at the edge of a film of non-Newtonian liquid.

The coordinate and velocity of motion of the edge of the film x_0, $v_0 = dx_0/dt$ remain indeterminate functions of the time, within the context of the foregoing; they may be found by accurate solution of the complete problem for h. However, some estimates may be made without the need for accurate solution, in the case of a specified liquid flux

$$Q(t) = \int_0^{x_0} dx\, x^{p-1} h = Q_0 t^q , \tag{3}$$

arriving at a symmetrically spreading film (p = 1 and 2 with plane and cylindrical symmetry). Using the edge asymptote in Eq. (2) for approximate description of the form of the whole film, a differential equation for h is obtained from Eq. (3), with the solution

$$\begin{gathered} x_0 = \xi_0 \theta^\beta , \; v_0 = \beta \frac{x_0}{t} , \; \theta = at, \; a^\mu = \frac{g}{k} Q_0^{n+2}, \\ \xi_0^{-\nu} = (n+2)\left(\frac{2n+1}{n}\beta\right)^n B^{n+2}\left(p, \frac{n+3}{n+2}\right), \; \beta = \frac{\mu}{\nu} , \\ \mu \equiv n + q(n+2), \; \nu \equiv n + 1 + p(n+2). \end{gathered} \tag{4}$$

This approximate result is in good agreement with the consequences of accurate self-similar solution of Eqs. (6) and (7). For example, when q = 0, n = p = 1, it follows that $\xi_0 \simeq 1.06$, whereas from the accurate solution $\xi_0 \simeq 1.13$ (a difference in the third figure!).

The parameter β characterizing the variation in x_0 over time is found to be larger for dilatational liquids (n > 1), and smaller for pseudoplastic liquids, than for Newtonian liquids, when q < 1 + 2p. When z < 1 + 2p, the situation is reversed. When q < p + 1/(n + 2), β is less than one, and the velocity of motion of the film edge falls with time (more slowly for pseudoplastic media).

The initial problem of the spreading of a film with h = 0 when t = 0, x > 0, with a specified liquid flux of the type in Eq. (3) at the coordinate origin, has a self-similar solution $h = t\Phi(x/t^2)$ for an arbitrary non-Newtonian viscous liquid, if q = 1 + 2p. However, this solution is only of limited interest, generally speaking. Its characteristic spatial scales are $H \sim t$, $L \sim t^2$, and the dimensionless parameters vary as follows: $\varepsilon \sim 1/t$, $\varepsilon \mathrm{Re} \sim t/\nu^2$. Consequently, the inertial dimensionless parameter εRe increases over time even for an ordinary viscous liquid, and in a finite time the noninertial condition used in deriving Eq. (1) for the film thickness no longer holds.

More meaningful self-similar solutions of the given initial problem with the condition in Eq. (3) are possible in the particular case of a nonlinear liquid for which Eq. (1) takes the form

$$\frac{\partial h}{\partial t} = \frac{n}{2n+1}\left(\frac{g}{k}\right)^{1/n} \nabla\left(h^{2+1/n} |\nabla h|^{1/n-1} \nabla h\right). \tag{5}$$

This equation coincides with the "equation of turbulent filtration," various self-similar solutions of which were analyzed in [3].

The self-similar distribution

$$h = \frac{Q_0}{a^q} \theta^\alpha \varphi(\xi), \; \xi = \frac{x}{\theta^\beta}, \; \theta = at, \tag{6}$$

is a solution of Eq. (5) with the condition in Eq. (3), if β and the dimensional multiplier a are determined from Eq. (4) and α from the formula

$$\alpha = q - \beta p = \frac{(n+1)q - np}{n + 1 + p(n+2)} ,$$

while the dimensionless function $\varphi(\xi)$ satisfies the equation

$$q\xi^{p-1}\varphi = \frac{\partial}{\partial\xi}\left(\beta\xi^p\varphi + \frac{n}{2n+1}\xi^{p-1}\varphi^{2+1/n}\left|\frac{\partial\varphi}{\partial\xi}\right|^{1/n}\mathrm{sgn}\frac{\partial\varphi}{\partial\xi}\right),$$

$$\int_0^{\xi_0} d\xi\,\xi^{p-1}\varphi(\xi) = 1.$$

For the self-similar solution with the given α, β, the dimensionless parameter ε always decreases over time, while the product $\varepsilon\,\mathrm{Re}$ decreases only when $q < q_0 \equiv p + (p + 2)/(n + 3)$. Under this condition, the spreading of the liquid is asymptotically ($t \to \infty$) noninertial. In the opposite case $q > q_0$ (for a Newtonian liquid, $q = 7/4$ and 3 with one-dimensional and axisymmetric spreading, respectively), noninertial behavior is possible only for a finite time.

In the absence of an external liquid flux ($q = 0$), when its volume is conserved, the equation for (ξ) admits of simple finite ($\varphi = 0$ when $\xi \geq \xi_0$) analytic solution (cf. [2, 4])

$$\varphi(\xi) = C\left(\xi_0^{n+1} - \xi^{n+1}\right)^{\frac{1}{n+2}}, \quad \xi \leqslant \xi_0;$$

$$C^{n+2} = \frac{n+2}{n+1}\left(\frac{2n+1}{\nu}\right)^n, \quad \xi_0^{\frac{\nu}{n+2}} B\left(\frac{p}{n+1}, \frac{n+3}{n+2}\right) = \frac{n+1}{C}.$$

Since $q = 0 < q_0$ here, this function corresponds to the asymptotically noninertial solution in Eq. (6).

2. Constrained Spreading over a Smooth Horizontal Surface

Now consider the spreading of a nonlinearly viscous liquid over a horizontal plane when there are bounding vertical side walls. In the case which is simplest for analysis — a narrow gap between walls with $d \equiv \delta H \ll H$; as before, the film is assumed to be thin: $H \equiv \varepsilon L \ll L$ — the horizontal shear stress $\tau = \tau_{xy}$ due to the friction at the vertical walls takes on the principal role. The asymptotically simplified (as in Sec. 1, but with two small parameters ξ, δ) equations and boundary conditions take the following form, neglecting inertial forces ($\varepsilon\delta^2\,\mathrm{Re} \ll 1$) and surface tension ($\mathrm{We} \ll 1$)

$$\frac{\partial p}{\partial x} = \frac{\partial\tau}{\partial y}, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -g, \quad \tau = \eta\left(\left|\frac{\partial u}{\partial y}\right|\right)\frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (p - p_a)|_{z=h} = 0, \quad \frac{\partial h}{\partial t} + \left(u\frac{\partial h}{\partial x} - w\right)\Big|_{z=h} = 0,$$

$$0 = u|_{y=0} = u|_{y=d} = u|_{z=0} = \cdots$$

hence the stress and velocity may be expressed in terms of the film thickness $h(x, t)$ as follows, where $\langle\ldots\rangle$ denotes averaging over the gap between the vertical walls

$$\tau = -g\frac{\partial h}{\partial x}\left(\frac{d}{2} - y\right), \quad \tau_v = \tau\Big|_{y=0},$$

$$u = \left[\left|\frac{d}{2} - y\right|F_0(|\tau|) - \frac{d}{2}F_0(|\tau_v|)\right]\mathrm{sgn}\frac{\partial h}{\partial x},$$

$$\langle u\rangle = -\frac{d}{2}F_1(|\tau_v|)\,\mathrm{sgn}\frac{\partial h}{\partial x}$$

and the following equation is obtained for $h(x, t)$

$$\frac{\partial h}{\partial t} = \frac{d}{2}\frac{\partial}{dx}\left(hF_1(|\tau_v|)\,\mathrm{sgn}\frac{\partial h}{\partial x}\right), \quad |\tau_v| = \frac{gd}{2}\left|\frac{\partial h}{\partial x}\right|. \tag{8}$$

Close to the edge of the spreading film, after replacing $\partial h/\partial t$ by $-v_0\partial/\partial x$, the equation is integrated to give

$$F_1(|\tau_v|) = \frac{2v_0}{d}, \quad |\tau_v| = -\frac{gd}{2}\frac{\partial h}{\partial x},$$

which shows that, for an arbitrary non-Newtonian liquid close to the edge of the film ($x_0 - x \ll x_0$) in a narrow gap, the shear stress does not change with increasing distance from the edge, and the film thickness varies linearly

$$h \approx \gamma (x_0 - x), \quad x \leqslant x_0. \tag{9}$$

In contrast to the case of an infinite liquid, the free surface at the edge of the film in a gap forms an acute angle with the horizontal substrate.

The initial problem with h = 0 when t = 0, x > 0, for Eq. (8), in the case where there is an incoming flux as in Eq. (3) at the coordinate origin, admits of self-similar solution of the form $h = t\Phi(x/t)$ for any non-Newtonian liquid, if q = 2. It is readily evident that the linear distribution in Eq. (9) with $x_0 = v_0 t$, v_0 = const is an accurate solution if $\gamma v_0^2 = 2Q_0$ and $2v_0 = dF_1(\gamma g d/2)$. For the given self-similar solution H ~ t, L ~ t, and hence $\varepsilon \sim t^0$, $\delta \sim 1/t$, $\varepsilon\delta^2 \mathrm{Re} \sim 1/(t\nu^2)$. In contrast to the case of nonconstrained spreading over a plane (Sec. 1), the noninertial condition ($\varepsilon\delta^2 \mathrm{Re} \ll 1$) is asymptotically (as $t \to \infty$) true here for a large class of liquids (for a Newtonian liquid with constant viscosity, in particular).

For a nonlinear liquid, Eq. (8) reduces to the equation

$$\frac{\partial h}{\partial t} = \frac{n}{2n+1}\,\frac{d}{2}\left(\frac{gd}{2k}\right)^{1/n}\frac{\partial}{\partial x}\left(h\left|\frac{\partial h}{\partial x}\right|^{1/n}\operatorname{sgn}\frac{\partial h}{\partial x}\right) \tag{10}$$

and under the condition in Eq. (3) – for a Newtonian liquid with n = 1, this problem coincides with that analyzed in [5] – has a self-similar solution of the form in Eq. (6) with the coefficients

$$\alpha = \frac{q(n+1)-n}{n+2}, \quad \beta = \frac{q+n}{n+2}, \quad a^{n+q} = \frac{g}{k}\, d^{n+1} Q_0 \tag{11}$$

and a dimensionless function $\varphi(\xi)$ satisfying the relations

$$q\varphi = \frac{\partial}{\partial \xi}\left(\beta\xi\varphi + \frac{n}{2n+1}\, 2^{-1-1/n}\,\varphi\left|\frac{\partial\varphi}{\partial\xi}\right|^{1/n}\operatorname{sgn}\frac{\partial\varphi}{\partial\xi}\right), \quad \int_0^{\xi_0} d\xi\,\varphi(\xi) = 1.$$

In the particular case when q = 0, the latter relation admits of simple analytical solution

$$\varphi(\xi) = C(\xi_0^{n+1} - \xi^{n+1}), \quad \xi \leqslant \xi_0,$$

$$C = \frac{2}{n+1}\left(2\,\frac{2n+1}{n+2}\right)^n, \quad (2n+1)^n \xi_0^{n+2} = \left(\frac{n+2}{2}\right)^{n+1}, \tag{12}$$

in which the liquid spreads with a mean (over the gap) velocity $\langle u \rangle = nx/t(n+2)$ and at the edge of the film

$$h \approx \left(2v_0\,\frac{2n+1}{n}\right)^n \frac{2k}{gd^{n+1}}(x_0 - x), \quad v_0 = \frac{n}{n+2}\,\frac{x_0}{t}.$$

[cf. Eq. (9)].

For the given self-similar spreading of a film of nonlinear liquid, $H \sim t^{\alpha}$, $L \sim t^{\beta}$ with α and β as in Eq. (10), and $\varepsilon \sim t^{\alpha-\beta}$, $\delta \sim t^{-\alpha}$, $\varepsilon\delta^2 \mathrm{Re} \sim \delta\varepsilon^{2/n}$. If the small parameters ε, δ are not to increase asymptotically (as $t \to \infty$), it is necessary that $2 \geq q \geq n/(n+1)$ (under this condition, the inertial parameter $\varepsilon\delta^2 \mathrm{Re}$ is also small). The limit q = 2 corresponds to the above-discussed solution of form $t\Phi(x/t)$. The analytical solution in Eq. (12) obtained when q = 0 does not satisfy the given condition. For such solutions – and, in general, for solutions with $q < n/(n+1)$ – decrease in film thickness leads to increase in δ over time. Ultimately, it is found that the condition $d = \delta H \ll H$ used in deriving the above simplified equations for the film thickness does not hold.

NOTATION

u, v, w, velocity components; u, horizontal velocity vector; ∇, horizontal-gradient operator; η, viscosity; p, pressure; g, acceleration due to gravity; n, exponent of flow power law; k, consistency of liquid; Re, Reynolds number; ε = H/L; δ = d/H, small parameters; We, Weber number; h, thickness of liquid film; x, horizontal coordinate vector; z, vertical coordinate; τ, shear stress; $F_\ell(\tau)$, integral moments of rheological function $f(\tau)$; τ_w, stress at horizontal substrate; τ_v, stress at vertical wall; x_0, v_0, coordinate and velocity of motion of film edge; Q(t), liquid influx; α, β, self-similarity exponents; $\varphi(\xi)$, dimensionless function of dimensionless argument; B(a, b), beta function; C, numerical constant; d, distance between vertical planes.

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SPREADING OF SMALL LIQUID DROPS ALONG A FLAT SURFACE

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We discuss two limiting spreading laws for small drops of a viscous liquid which are supported by the experimental data.

Drops of coolant are sprayed onto a surface in order to obtain "soft" cooling. Calculation of the effectiveness of the heat transfer in this case is not possible without a knowledge of the processes of wetting and spreading of the drops adhering to the surface. Because of their small size, one can assume that the drops are isothermal with a variable temperature.

A phenomenological method of describing the spreading of a partially wetting liquid in the viscous regime was given in [1]. In this method an additional body force (equal to the gradient of the chemical potential) is introduced into the equations of hydrodynamics. For an incompressible liquid under isothermal conditions [2] this approach is equivalent to introducing an additional "disjoining" pressure [3, 4].

The equations of motion of a viscous liquid, written in the approximation of the theory of lubrication [5], together with the boundary conditions on the surface of the liquid drop, can be solved for the function h(x, y, t) determining the shape of the free surface [6] [x and y are Cartesian coordinates in the plane of the solid surface z = 0; t is the time; the shape of the free surface of the liquid is given by the equation z = h(x, y, t)].

We assume that the shape of the drop is axially symmetric. With no loss of generality, we can take the line x = y = 0 as the symmetry axis. Let $a(t)$ be the radius of the circle wetted by the liquid on the solid surface z = 0. Then the equation of the line of three-phase contact is $x^2 + y^2 = a^2$. For further analysis it will be convenient to introduce polar coordinates: $r^2 = x^2 + y^2$, $\varphi = \mathrm{arctg}\,(x/y)$. By symmetry h = h(r, t). If we then introduce the "local" coordinate $x^* = a(t) - r$, we find that close to the surface of the drop ($x^* \to +0$) the equation for the shape of the free surface reduces to the corresponding equation for the "plane" case considered in [1], which can be written (in dimensionless variables)

$$\frac{1}{2}\,\frac{\partial^2\beta}{\partial\eta^2} + R\,\frac{\partial}{\partial\eta}\,[\Phi_0(\eta,\ \beta)] + \frac{u}{\alpha\eta^2} = 0,\quad u \equiv \frac{da^*}{dt^*}. \tag{1}$$

Here $L = \sqrt{\sigma/\rho g}$ is taken as the unit of length; $T = 3\ \mu L/\sigma$ is the unit of time; $\beta = \alpha^2 - \theta^2 \geq 0$, $\alpha(\eta,\ t) \geq 0$ is the angle of inclination of the surface of the drop to the solid surface (in the approximation considered here $\alpha^2 \approx (\partial h/\partial x^*)^2$); θ is the equilibrium value of the

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